

Generalized statistics and the algebra of observables*

Jon Magne Leinaas
 Centre for Advanced Study,
 Drammensveien 78,
 N-0271 Oslo, Norway
 and
 Department of Physics, University of Oslo,
 P.O. Box 1048 Blindern,
 N-0316 Oslo 3, Norway.[†]

Abstract

A short review is given of how to apply the algebraic Heisenberg quantization scheme to a system of identical particles. For two particles in one dimension the approach leads to a generalization of the Bose and Fermi description which can be expressed in the form of a $1/x^2$ statistics interaction between the particles. For an N-particle system it is shown how a particular infinite-dimensional algebra arises as a generalization of the $su(1,1)$ algebra which is present for the two-particle system.

1 Introduction

In recent years the possibilities of introducing generalized types of particle statistics have attracted a great deal of attention. Although there is no reason to doubt the traditional classification of elementary particles into bosons and fermions, the investigation of natural generalizations of these two classes of particles is an interesting exercise from a theoretical point of view. On one side the investigation shows in what way bosons and fermions are special. On the other side such generalizations may be relevant for particles that are not elementary. This has been demonstrated in the case of the (fractional) quantum Hall effect, where the quasi-particle excitations are believed to have anyonic properties. (For an introduction to anyons and generalized statistics, see for example [1] in the proceedings of the summer school of 1992.)

There are several different approaches possible when discussing generalizations of bosons and fermions. I will loosely divide them into three classes. The first one may be referred to as the *quantum mechanics* approach. This approach addresses the question of how to quantize a system of identical particles in such a way that the indistinguishability of the particles is taken care of in the proper way [2, 3]. It is well known that such an analysis leads most

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[†]Permanent address.

naturally to consider *anyons* as a generalization of bosons and fermions in two-dimensional systems [3, 4, 5]. The second one is the *statistical mechanics* approach. In this approach one considers generalizations of the statistical distributions associated with bosons and fermions [6, 7, 8]. Thus, one considers generalizations of the Pauli exclusion principle in the form of modified rules for filling single particle levels. The third one is the *second quantization* approach. Here one considers the question of possible generalizations of the commutation and anti-commutation relations between field operators in the cases of bosons and fermions respectively [9, 10, 11]. These three approaches are not equivalent, although in some cases they lead to the same type of generalizations. I will here only discuss the quantum mechanics approach.

There are in fact (at least) two different ways, which both seem natural, to approach the question of how to properly quantize a system of identical particles. The first one is the *Schrödinger* approach, which leads to the concept of anyons. In this approach one focuses the attention on the configuration space of the system and introduces wave functions which are defined on this space [3]. What is special about the configuration space for a system of identical particles is that the N -particle space is not the product space of N single particle spaces. Instead the correct configuration space is derived from the product space by introducing identifications between points which correspond to reordering of the particles. The result is a space with singularities, and there is a phase factor associated with the wave functions when a singularity is encircled. Such an encircling is physically associated with the interchange of the position of two particles, and the corresponding phase factor is $+1$ for bosons and -1 for fermions. In two space dimensions other phase factors are conceivable and intermediate (anyonic) types of statistics is a possibility. (The path integral formulation may be regarded as a distinct approach, but also this is based on the correct description of the configuration space, and it essentially gives the same result as the Schrödinger quantization [2, 12].)

The second approach (*Heisenberg* quantization) focuses the attention on the observables rather than the wave functions. Quantization is introduced in the form of a set of fundamental commutation relations between observables. Also here the indistinguishability of the particles interferes in a fundamental way with the quantization of the system. The main point is that the observables are symmetric with respect to the interchange of particles and the fundamental commutation relations are changed as compared with a system of non-identical particles, where the observables are non-symmetric. As a consequence of this, new possibilities appear and these are interpreted as generalized statistics. A somewhat surprising result is that the Heisenberg and Schrödinger approaches give rise to non-equivalent generalizations of bosons and fermions [13, 14].

It is the algebraic Heisenberg quantization scheme which will be discussed here. Quantization is formulated as a program for finding irreducible representations of a fundamental Lie algebra of observables. The program will first be illustrated by two simple examples and the system of two identical particles in one dimension will then be discussed in some detail. Different types of statistics will be shown to correspond to inequivalent representations of the same fundamental algebra. It is argued that the *Calogero* model [15], *i.e.* a system of particles with $1/x^2$ interaction, can be viewed as a coordinate description of these generalized statistics. Some of the peculiarities of this system are briefly discussed and the question is addressed whether there also for the N -particle system is some underlying fundamental algebra, such that different values of the statistics parameter correspond to inequivalent representations of this algebra. Such a parameter independent algebra has indeed been shown to be

present in the Calogero model [16]. I will discuss how this infinite-dimensional algebra can be constructed by use of operators from the Calogero model and show that the algebra can be defined in a more abstract way. This re-formulation of the algebra makes it possible to identify the presence of the same algebra in another, but related system, the matrix model. In this model the variables are $N \times N$ Hermitian matrices and the model can be related to the Calogero model by fixing some of the constants of motion. I conclude with some comments about unsolved problems and possible physical applications of the algebra.

The Heisenberg quantization scheme for identical particles, which is discussed here, has been developed together with Jan Myrheim [13, 14]. The infinite-dimensional algebra of the Calogero model has been studied in a paper with Serguei Isakov [16], and finally the simplifications in the formulation of the algebra and the application to the matrix models have been discussed in a work with Jan Myrheim, Serguei Isakov, Alexios Polychronakos and Raimund Varnhagen [17].

2 The algebraic Heisenberg quantization scheme

In Heisenberg's formulation of quantum mechanics the commutation relation between position and momentum is the basic ingredient [18]. It *defines* the quantum mechanics of the system in the sense that it can be used to derive the matrix elements of any observable, *i.e.* any function of position and momentum. Heisenberg's canonical commutation relation can be viewed as defining a particular Lie algebra. This Lie algebra is represented in the classical description of the system in the form of Poisson brackets as well as in the quantum description in the form of commutators. The quantum mechanical system can be seen as defining a representation of this fundamental *Lie algebra of observables* in terms of a commutation algebra of Hermitian operators in a Hilbert space.

Viewed in this way we may formulate the Heisenberg approach to quantum mechanics as a general algebraic program for the quantization a classical system. The program has the following ingredients:

- a) Choose a complete set of fundamental observables A, B, \dots which form a closed Lie algebra under Poisson brackets.
- b) Represent the observables as Hermitian operators \hat{A}, \hat{B}, \dots which define an irreducible representation of the algebra, with Poisson brackets replaced by commutators, $\{A, B\}_{pb} \rightarrow \frac{1}{i\hbar} [\hat{A}, \hat{B}]$.
- c) Represent a general observable as a function of the fundamental observables \hat{A}, \hat{B}, \dots .

Since the representation is irreducible (which reflects the fact that the set of observables is complete), the eigenvalues and matrix elements (in a chosen basis) of all the observables are determined by their algebraic relations. In the case of the (Heisenberg-Weyl) algebra of position and momentum, the irreducible representation is unique. More generally we expect that several, inequivalent representations may exist. These different representations can be interpreted as inequivalent quantizations of the same classical system. As I will discuss, the different particle statistics appear in this approach as inequivalent representations of the same algebra of observables for a system of identical particles.

Clearly the program for quantization referred to above is somewhat loosely defined. There

are several remaining questions and possible ambiguities. One possible ambiguity has to do with the choice of a fundamental set of observables. Different choices may lead to different results. Another ambiguity has to do with the operator ordering problem. This makes the mapping from the classical to the quantum observables non-unique. It may also affect the fundamental algebra itself in the form of quantum corrections to the commutation relations. And finally there may be a conflict between the number of fundamental observables which is needed to close the Lie algebra and the number of independent variables of the (classical) system.

It is not my intention to make any attempt to give a more precise formulation of the general program. After all we have to accept that there are inherent ambiguities in the quantization of a classical system. However, for sufficiently simple systems the problems and ambiguities may be less important. I will first demonstrate this for two simple cases of a single particle, and then turn to the question of what happens for a system of identical particles in one dimension.

3 Systems of non-identical particles

For a system of non-identical particles there is an independent set of observables for each particle of the system. This means that the N -particle (Hilbert) space is a product space of single particle spaces. The quantization of the system therefore can be accomplished by quantizing the single particle systems separately. Interactions may certainly make the dynamics of the N -particle system complicated, but this we do not see as important for the question of how to quantize the system. (Note, however, that this view may be too simpleminded if there are singular interactions or velocity-dependent forces present.) With this argument in mind, I will briefly examine two elementary systems where a single particle moves in one dimension, first on an infinite line and then on a circle.

3.1 One particle on a line – the Heisenberg-Weyl algebra

This first example is almost too simple. It is well-known from introductory quantum mechanics and it is the original case discussed by Heisenberg, Born and Jordan [18] in the form of a fundamental commutation relation between position and momentum. However, since this is the model for what I will also later discuss, I will nevertheless include a brief discussion.

The natural set of fundamental observables in this case consists of the position x and the momentum p . If we extend this by a third observable, which I will denote λ , we have a closed Lie algebra,

$$\begin{aligned} [x, p] &= i\lambda, \\ [x, \lambda] &= [p, \lambda] = 0 \end{aligned} \tag{1}$$

In an irreducible representation λ is proportional to the identity, and the standard form of the canonical quantization condition is obtained if we write it as

$$\lambda = \hbar \mathbf{1} \tag{2}$$

(Note that other values of λ do not correspond to different representations, since they can be related by a rescaling of x and p .)

An irreducible representation of the algebra is most readily constructed in the standard way by use of the raising and lowering operators

$$\begin{aligned} a &= \frac{1}{\sqrt{2\mu}} \left(x + \frac{i}{\hbar} \mu p \right) \\ a^\dagger &= \frac{1}{\sqrt{2\mu}} \left(x - \frac{i}{\hbar} \mu p \right) \end{aligned} \quad (3)$$

where μ is an arbitrary parameter of the correct dimension. Hermiticity requires a lowest state annihilated by a

$$a|0\rangle = 0 \quad (4)$$

and a complete set of basis states is constructed iteratively by use of a^\dagger ,

$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle \quad (5)$$

The construction shows that the irreducible representation is unique.

The connection to the Schrödinger formulation is obtained if we introduce a orthonormalized set of position states, the eigenstates of x . In this coordinate representation p is represented, in the usual way, as the differential operator $\frac{\hbar}{i} \frac{d}{dx}$.

As already mentioned the generalization to several particles, and also to higher dimensions, is straight forward, since there is an independent algebra for each particle and each dimension.

3.2 One particle on a circle

If the particle moves on a circle rather than on a straight line, it is well known from the Schrödinger approach that a new feature appears. The wave functions will pick up a complex phase factor when the circle is traversed. This phase factor is not specified by the classical theory, and different values correspond to inequivalent quantizations of the system. A physical interpretation is that the particle is charged and that the complex phase is related to a magnetic flux which penetrates the circle.

Let us examine the system from the algebraic point of view. We first note that the polar angle should not be considered as an observable for the position of the particle. This is because it is not invariant when the particle completes a full cycle. Let us instead use the cartesian coordinates x and y as observables for the position. Together with the angular momentum L they form a closed Lie algebra,

$$[L, x] = i\hbar y \quad [L, y] = -i\hbar x \quad [x, y] = 0 \quad (6)$$

Apparently we have too many observables to describe the two-dimensional phase space, but there is an invariant, which in an irreducible representation takes a constant value,

$$x^2 + y^2 = r^2 \mathbf{1} \quad (7)$$

The constant r is naturally interpreted as the radius of the circle and the relation reduces the number of independent variables to two.

Also in this case the irreducible representations are most conveniently constructed by use of raising and lowering operators, but now we choose $x_\pm = x \pm iy$. They satisfy

$$[L, x_\pm] = \pm \hbar x_\pm \quad (8)$$

and therefore are climbing operators in the spectrum of L . Beginning from an arbitrary eigenstate of L , a tower of eigenstates is constructed, which span the Hilbert space,

$$\begin{aligned} L|m\rangle &= (m + \alpha)\hbar|m\rangle \\ x_{\pm}|m\rangle &= r|m \pm 1\rangle \end{aligned} \quad (9)$$

The parameter m takes all integer values, and the spectrum of L is unbounded both from above and below. The parameter α , which describes a shift of the spectrum relative to the integer values is undetermined by the algebraic relations. It characterizes inequivalent representations of the algebra.

A coordinate representation can be introduced in terms of the eigenstates of the position operators x_{\pm} and the corresponding wave functions,

$$\begin{aligned} x_{\pm}|\phi\rangle &= re^{\pm i\phi}|\phi\rangle \\ \psi(\phi) &= \langle\phi|\psi\rangle \end{aligned} \quad (10)$$

The form of the angular momentum operator in the coordinate representation can be deduced from the commutation relations. If we choose the position eigenvectors and therefore the wave functions to be single valued on the circle, the operator is found to be

$$L = -i\hbar \left(\frac{d}{d\phi} + \alpha \right) \quad (11)$$

Clearly this result agrees with the Schrödinger quantization of the system. The parameter α in that case appears as through the periodicity condition on the wave functions. In the present case the parameter defines inequivalent irreducible representations of the same fundamental algebra of observables.

4 Identical particles and the $1/x^2$ interaction

For identical particles there is a restriction on the observables relative to the case of non-identical particles. The observables should be symmetric with respect to particle indices. We consider first the implication of this for the case of two identical particles on the (infinite) line [13]. The center-of-mass coordinate $X = \frac{1}{2}(x_1 + x_2)$ and momentum $P = p_1 + p_2$ are not affected by the symmetry constraint, and they define the same Heisenberg-Weyl algebra as that of a single particle. However, the constraint is important for relative coordinate and momentum. The linear variables $x = x_1 - x_2$ and $p = \frac{1}{2}(p_1 - p_2)$ are not observables since they are not symmetric, but the quadratic variables are, and they form a closed Lie algebra. If we choose the following combinations

$$A = \frac{1}{4}(p^2 + x^2) \quad B = \frac{1}{4}(p^2 - x^2) \quad C = \frac{1}{4}(px + xp) \quad (12)$$

and introduce the Hermitian conjugate operators

$$B_{\pm} = B \pm iC \quad (13)$$

these operators satisfy the standard commutation relations of the $su(1,1)$ or $sp(2, R)$ algebra

$$[A, B_{\pm}] = \pm B_{\pm} \quad [B_+, B_-] = -2A \quad (14)$$

(In these expressions I have assumed dimensionless variables x and p .)

The quadratic variables A , B and C (together with the CM variables X and P) may be considered as the fundamental observables of the system. Following the general approach outlined above we now quantize the system by constructing irreducible representations of the fundamental algebra of observables. (In the following we suppress the CM coordinate and momentum since they can easily be handled as in the single particle case.)

The representations can be constructed in the same way as for the particle on the circle. We construct a tower of eigenstates of the operator A by use of the lowering and raising operators B_- and B_+ . Hermiticity in this case implies that there is a lowest state annihilated by B_- , and for general eigen vectors of A we find the relations

$$\begin{aligned} B_- |n\rangle &= \sqrt{n(n-1+2\alpha)} |n-1\rangle \\ B_+ |n\rangle &= \sqrt{(n+1)(n+2\alpha)} |n+1\rangle \\ A |n\rangle &= (\alpha+n) |n\rangle \end{aligned} \tag{15}$$

These relations define an infinite basis for the representation and specify the matrix elements of the fundamental observables. We note that again a (real) parameter α appears, which characterizes inequivalent representations of the algebra. With n restricted to non-negative integers, the parameter α is restricted (by Hermiticity of the observables) to be positive. Different values of α we interpret as corresponding to inequivalent quantizations of the two-particle system.

The spectrum of A , which is identical to a harmonic oscillator spectrum, shows that the parameter value $\alpha = 1/4$ corresponds to a system of bosons, while $\alpha = 3/4$ corresponds to fermions. One may note that there is no exact periodicity in α , but there is a quasi-periodicity in the sense that all levels except the lowest one are reproduced when α increases by 1.

Also in this case a coordinate representation may be introduced in terms of eigenvectors of the operator x^2 . From the action of the operators on these states, one finds expressions for the observables A , B and C in terms of differential operators which are unusual in the sense that the operator p^2 includes a singular potential,

$$p^2 = -\frac{d^2}{dx^2} + \frac{\lambda}{x^2}, \tag{16}$$

where strength of the potential λ is determined by the parameter α ,

$$\lambda = 4 \left(\alpha - \frac{1}{4} \right) \left(\alpha - \frac{3}{4} \right) \tag{17}$$

This means that the kinetic energy of the two-particle system in the general case will include a “statistical” $1/x^2$ -potential. This potential vanishes only for bosons and fermions.

Before proceeding, I would like to include some comments about the $1/x^2$ potential. There are several features of a system of particles with this interaction which are similar to those of a free particle system (see ref.[19] for a detailed discussion). Let us first consider a two-particle collision. A characteristic feature of this collision is that there is no time delay. This means that the particles after the collision (asymptotically) move as if they just passed through each others. Only at close distance the particle trajectories deviate from those of non-interacting

particles. In the quantum description of the collision one finds that there is a constant phase shift

$$\phi_{sc} = -2\pi \left(\alpha - \frac{1}{4} \right) \quad (18)$$

The momentum independence of this phase shift is directly related to the lack of time delay in the collision.

These features in fact carry over to the N -particle system. The effect of scattering of the classical system is just to permute the momenta of the incoming particles in the out state, and there is no time delay relative to the motion of non-interacting particles. In the quantum case the overall phase shift is again momentum independent and it is simply the sum of two-particle contributions (18) from each pair of particles in the system.

Let us next consider what happens when a harmonic oscillator interaction is added to the kinetic term. For two particles we already know the result, since the Hamiltonian is essentially the observable A of the $su(1,1)$ algebra. The effect of the $1/x^2$ -interaction is simply to shift all the energy levels so that the constant separation between the levels is preserved. For the N -particle system the Hamiltonian gets the form

$$H = \frac{1}{2} \sum_{i=1}^N \left(-\frac{\partial^2}{\partial x_i^2} + 2 \sum_{i \neq j} \frac{\lambda}{(x_i - x_j)^2} + x_i^2 \right) \quad (19)$$

This interacting model is known as the Calogero model and it has been extensively studied since it was first presented and shown to be exactly solvable [15]. Again the effect of the interaction is to give an overall shift of the whole energy spectrum, so that the energies can be written in the form

$$E = E_B + \left(\alpha - \frac{1}{4} \right) N(N-1) \quad (20)$$

where E_B is the corresponding energy of the bosonic N -particle harmonic oscillator. (The energy is here scaled with the harmonic oscillator frequency to be a dimensionless quantity.)

5 An algebra for more than 2 particles

So far I have restricted the discussion of the fundamental algebra of observables to the case of two identical particles in one dimension. The generalization to higher dimensions (but still only two particles) is straight forward in the sense that it is easy to identify the algebra. The quadratic variables in the relative coordinate and momentum define the Lie algebra $sp(2d, R)$, where d is the dimension of the one-particle configuration space. These variables generate the linear symplectic transformation in the $2d$ -dimensional phase space. Representations of these algebras may be constructed in a similar way as discussed here for $d = 1$, but the construction becomes more complicated with additional degeneracies of the energy levels. With Jan Myrheim I have examined the case of two particles in two dimensions in some detail. Also in this case we find representations different from those which describe bosons and fermions, but the interpretation is not as simple as in one dimension, since additional degrees of freedom appear in the general case. I will only refer to [14] for a discussion of the

two-dimensional case, while here returning to the question of the algebra of observables for a system of identical particles in one dimension ¹.

When the number of particles is larger than 2 the symmetric polynomials in x and p of degree lower or equal to 2 do not form a complete set of observables. There are symmetric functions of x and p that cannot be expressed in terms of these variables alone. Higher order polynomials are needed, but may be restricted to a degree lower or equal to N . However, the problem then arises that these polynomials do not form a closed finite-dimensional Lie algebra. We have the choice either to consider an algebra which is non-linear (*i.e.* products of the generators will appear as the result of commutation), or we extend the algebra to an infinite-dimensional algebra by including polynomials of arbitrarily high degree while neglecting all algebraic relations between the polynomials that exist for a finite particle number N . The latter point of view is only the one which seems possible to handle and it is this view I will adopt in the following. More specifically the algebra is assumed to be spanned by polynomials of the one-particle form

$$L_{mn} = \sum_{i=1}^N x_i^m p_i^n \quad (21)$$

where in the quantum case the ordering of the operators becomes important. This set of variables is large enough to generate all (symmetric) observables in x and p . The philosophy is that the algebraic relations between these variables, which appear for finite N are not included in the definition of the algebra, but rather appear at the level of irreducible representations. In particular the value of the particle number will be specified at this level, since the algebra is independent of the number of particles in the system.

The classical Lie algebra is defined by the Poisson brackets, which have the form

$$\{L_{kl}, L_{mn}\}_{pb} = (kn - ml)L_{k+m-1, l+n-1} \quad (22)$$

We may interpret the variables L_{mn} as generators for canonical transformations in a two-dimensional phase space. The infinite-dimensional Lie algebra in fact generate the full group of (continuous) canonical transformations.

The canonical transformations are area-preserving, and for a system of particles this means that the transformations preserve the particle density. This has been noted as interesting for the description of the incompressible states of the quantum Hall system, where the electrons have effectively a two-dimensional phase space due to the strong magnetic field. In a classical description of this incompressible fluid the infinite-dimensional algebra has the effect of generating perturbations of the edge of the system [22]. The classical Lie algebra defined by the generators L_{mn} is sometimes referred to as w_∞ . (More precisely it may be referred to as the positive part of the algebra, which then is defined to include generators L_{mn} also for negative m, n .)

A precise definition of the quantum observables L_{mn} depends on a specification of the operator ordering of x and p in the product. However, a reordering gives an operator which can be expressed as a linear combination of operators with the original ordering. This means that we can pick an arbitrary ordering of operators for each m, n and view a reordering as a change of basis of the algebra which is defined by these operators.

¹Note that in [14] the notation $sp(d, R)$ is used for the algebra instead of $sp(2d, R)$.

The Lie algebra of the quantum observables are defined by the commutation relations between the operators L_{mn} . For bosons as well as for fermions they have the form

$$\begin{aligned} [L_{kl}, L_{mn}] = & i\hbar(km - ln)L_{k+m-1, l+n-1} \\ & + i\hbar^3 c_{klmn} L_{k+m-3, l+n-3} \\ & + \dots \end{aligned} \quad (23)$$

We note that there are quantum corrections (higher order in \hbar) relative to the classical algebra. The precise form of these correction terms depend on the choice of operator ordering, but the point is that these new terms cannot all be absorbed by a redefinition of the operators L_{mn} . The algebra of the quantum system, which is often referred to as (the positive part of) $W_{1+\infty}$ is then different from the classical algebra w_∞ .

I would now like to consider the question of generalizations of bosons and fermions in a similar way as already discussed for the 2-particle case. The fundamental algebra of observables then plays a central role, and the main point I will discuss is the question of how to identify this algebra. The generators will be assumed to be of the form L_{mn} . Clearly the classical algebra cannot be identical to this commutation algebra, since it does not even reproduce bosons and fermions. Another possibility would be to assume the algebra $W_{1+\infty}$ of the boson and fermion system to be the fundamental algebra and then to look for other representations to describe systems of particles with generalized statistics. Such generalized representations have already been considered in the context of the quantum Hall system [23]. However, a different view is adopted in [16] and will be discussed here. As already pointed out for the case of two particles, the generalized statistics found in the algebraic approach can be interpreted as a (statistical) $1/x^2$ interaction between the particles. This description of the two-particle system is naturally extended to a system of N particles and suggests that the Calogero model can be viewed as describing a system of identical particles with generalized (one-dimensional) statistics. Based on this view we make the assumption that the fundamental algebra is already represented in the Calogero model. This model can then be used to identify and to construct the algebra.

6 A parameter independent algebra for the Calogero model

As already noted for the 2-particle case there exist representations of the symmetrized quadratic polynomials in x and p that do not admit these observables to be expressed as functions of the single-particle (and non-observable) variables x_i and p_i , where i is the particle index. However, as shown in [20, 21] this is true only if we insist that x_i and p_j should satisfy the standard Heisenberg commutation relations. A special modification of this algebra may be introduced which depends on the (statistics) interaction parameter and where the observables are expressed as the standard symmetric sum over products of single-particle operators. Remarkably this algebra can be extended to an algebra for the full N -particle system and it defines a spectrum generating algebra for the Calogero Hamiltonian.

The modified Heisenberg algebra has been referred to as an S_N -extended Heisenberg algebra, since it includes permutation operators in addition to x and p . The form of the modified commutator is (with x and p again in dimensionless units)

$$[x_i, p_j] = i(\delta_{ij}(1 + \nu \sum_{k \neq i} K_{ik}) - \nu K_{ij}) \quad (24)$$

where K_{ij} are transposition operators that interchanges the particle indices i and j ,

$$K_{ij}x_j = x_iK_{ij} \quad K_{ij}p_j = p_iK_{ij} \quad (25)$$

and further satisfy a set of product relations characteristic for the permutation group.

The parameter ν which is present in the modified commutator (24) is related to the statistics parameter α which was introduced at an earlier stage, $\nu = 2\alpha - 1/2$. The strength of the statistical interaction in this new parameter is

$$\lambda = 4(\alpha - \frac{1}{4})(\alpha - \frac{3}{4}) = \nu(\nu - 1) \quad (26)$$

In the following I will use the parameter ν instead of α or λ .

To stay close to the notation of ref.[16] I will now change to the (Hermitian conjugate) variables

$$a_i = \frac{1}{\sqrt{2}}(x_i + ip_i) \quad a_i^\dagger = \frac{1}{\sqrt{2}}(x_i - ip_i) \quad (27)$$

instead of x_i and p_i and introduce the observables L_{mn} in a slightly changed way as

$$L_{mn}^\alpha \approx \sum_{i=1}^N a_i^{\dagger m} a_i^n \quad (28)$$

In this equation \approx means equal up to operator ordering. We use the additional index α to distinguish between operators which differ by orderings of the operators a_i and a_i^\dagger . It is the algebra of the observables L_{mn}^α which is the object of our interest, and by use of the (parameter-*dependent*) S_N -extended Heisenberg algebra I will sketch how a parameter-*independent* algebra of observables can be constructed. For further details I refer to [16].

Before we examine further this algebra of observables it is of interest to note that the Calogero Hamiltonian belongs to this algebra and is essentially identical to the operator L_{11} . (Actually, this is after a factor $\prod (x_i - x_j)^\nu$ has been separated out of the wave functions.) This operator has simple commutators with the operators a_i and a_i^\dagger ,

$$[H, a_i^\dagger] = a_i^\dagger \quad (29)$$

This implies that the eigenstates and eigenvalues of H can be constructed in a similar way as for the ordinary harmonic oscillator. Note however that when applying the non-symmetrized operators a_i^\dagger to build up the space of states one constructs an extension of the original Calogero model. The original model is defined on the subspace of symmetrized or of anti-symmetrized states. (For the same reason this extended model will define a reducible representation of the algebra of observables L_{mn}^α).

To define the algebra of observables I will follow the construction of ref.[16]. In this approach one defines rules for constructing the generators, and by repeatedly applying the rules an infinite-dimensional basis of generators is built up. Since the rules can be shown to generate ν -independent operators, the corresponding algebra will be parameter independent. Unfortunately this does not mean that the algebra can be expressed in a closed form, but rather that we have a method to construct a basis for the algebra step by step.

The basic building blocks of the construction are the variables

$$L_{m0} = \sum_i a_i^{\dagger m} \quad L_{0n} = \sum_i a_i^n \quad (30)$$

These operators are assumed to belong to the algebra and they are not affected by the operator ordering problem. (More precisely it is the Hermitian combinations of the operators that are elements of the algebra.) By commuting these operators we may now generate new elements of the algebra. If A_k denote an operator of the above type, we will after n steps produce a string of commutators of the form

$$S = [A_n, \dots, [A_3[A_2, A_1]] \dots] \quad (31)$$

(In the following I will refer to such a repeated commutator simply as a “string”.) The commutator of two strings can be expressed as a sum of other strings by use of the Jacobi identity. If therefore all strings are included, with the number n of operators arbitrarily large, the construction clearly defines an infinite-dimensional Lie algebra. This algebra is the one we seek.

There is an element we have to specify in this construction. If we calculate the commutators by repeated use of the fundamental commutation relation (24) we can bring any string into a standard form (*e.g.* normal ordered form). If all these expressions should be independent of ν (and K_{ij}), the resulting algebra would be identical to the algebra $W_{1+\infty}$ of bosons and fermions. However, by examining a few examples we can convince ourselves that all ν -dependent terms cannot be avoided if we introduce a unique operator ordering for a given product of a_i and a_i^\dagger . This means that the algebra $W_{1+\infty}$ is not represented in the Calogero model (at least not in this way).

To be able to proceed we have to make the basic assumption that operators L_{mn}^α that differ by reordering of operators a_i and a_i^\dagger should be considered as independent elements of the algebra. This means that when defining the (abstract) algebra we simply neglect ν -dependent relations that are present in the Calogero model. Again we make the assumption that such relations appear only at the level of irreducible representations. Some identities between reordered operators nevertheless have to survive to the level of the algebra. These are the ones needed to satisfy the Jacobi identity. Our precise definition of the algebra is that all ν -independent identities which are present in the Calogero model will be imposed on the elements of the algebra. Even if it is clear that an infinite-dimensional algebra can be constructed in this way, it is still a non-trivial question whether this algebra will be independent of the statistics parameter ν . However, that is in fact the case, as has been demonstrated in [16], and I will now outline how this can be shown.

Let us consider the commutator between an arbitrary observable of the form L_{kl}^α and an observable L_{0n} . It can be written in the form

$$[L_{0n}, L_{kl}^\alpha] = \sum_i \sum_j \sum_p \mathcal{A}_j^p [a_i^n, a_j^\dagger] \mathcal{B}_j^p \quad (32)$$

where \mathcal{A}_j^p and \mathcal{B}_j^p denote products of operators a_j and a_j^\dagger only referring to particle j . The commutator appearing in the expression on the right-hand side can be evaluated

$$[a_i^n, a_j^\dagger] = n\delta_{ij}a_i^{n-1} + \nu\delta_{ij} \sum_l \sum_{s=0}^{n-1} a_i^{n-1-s} a_l^s K_{il} - \nu \sum_{s=0}^{n-1} a_i^{n-1-s} a_j^s K_{ij}. \quad (33)$$

When this is inserted in the equation above, the ν -dependent terms vanish due to the summation over i and j . This important result we write as

$$\sum_{ij} \mathcal{A}_j [a_i^r, a_j^\dagger] \mathcal{B}_j = \sum_i \mathcal{A}_i r a_i^{r-1} \mathcal{B}_i, \quad (34)$$

This is the basic relation needed to show the ν -independence of the algebra. We note that the right-hand-side is a new one-particle operator. This implies that the commutator (32) can be written in the form

$$[L_{0n}, L_{kl}^\alpha] = \sum_\beta c_\beta^\alpha(n, kl) L_{k-1, n+l-1}^\beta \quad (35)$$

where the coefficients $c_\beta^\alpha(n, kl)$ are ν -independent. All ν -dependence has been absorbed in the different orderings of operators in the observables $L_{k-1, n+l-1}^\beta$.

Clearly a commutator which involves L_{n0} instead of L_{0n} can be written in a similar form. From this we conclude that a string of commutators of these two types of operators can be reduced to the form (35). This means that any string can be expressed as a linear combination of operators L_{kl}^α with ν -independent coefficients. Since the algebra has been defined to be spanned by this set of strings, this implies that algebra is independent of the statistics parameter ν .

7 A step by step construction

The situation then is that we know that a parameter independent algebra exists and that it can be constructed by repeated commutation of the basic elements L_{m0} and L_{0n} . This does not mean that we know the full structure of the algebra, but we are able to construct it in a stepwise fashion. To do the commutations is straight forward, but to check for identities between operators which correspond to reordered products of a_i and a_i^\dagger is non-trivial, since there is no obvious simple and systematic method to do this. The low-order part of the algebra has nevertheless been studied, in [16] by use of the operators of the Calogero model, and in [17] by use of the matrix representation which I will discuss later.

To classify the generators it is convenient to introduce the notion of a spin index. The three operators $\frac{1}{2}L_{01}$, $\frac{1}{2}L_{11}$ and $\frac{1}{2}L_{10}$ define an $su(1,1)$ subalgebra. It acts in the adjoint representation (*i.e.* by commutation) on the other generators of the algebra, and the full algebra can be divided into finite dimensional representations of the subalgebra. This means that the generators can be characterized by a spin quantum number. The raising and lowering operators $\frac{1}{2}L_{10}$ and $\frac{1}{2}L_{01}$ mix operators L_{mn} with the same value of $m+n$, while the commutator with $\frac{1}{2}L_{11}$ determines the “z-component” of the spin,

$$[\frac{1}{2}L_{11}, L_{mn}] = \frac{1}{2}(m-n)L_{mn} \quad (36)$$

By counting the number of states, one readily sees that $\frac{1}{2}(m+n)$ is the maximum spin associated with an operator L_{mn} . For the lowest values of $m+n$ the maximum spin multiplet is the only multiplet present, but for higher values of $m+n$ there are an increasing number of multiplets of lower spin. The number of multiplets can be determined from the list of

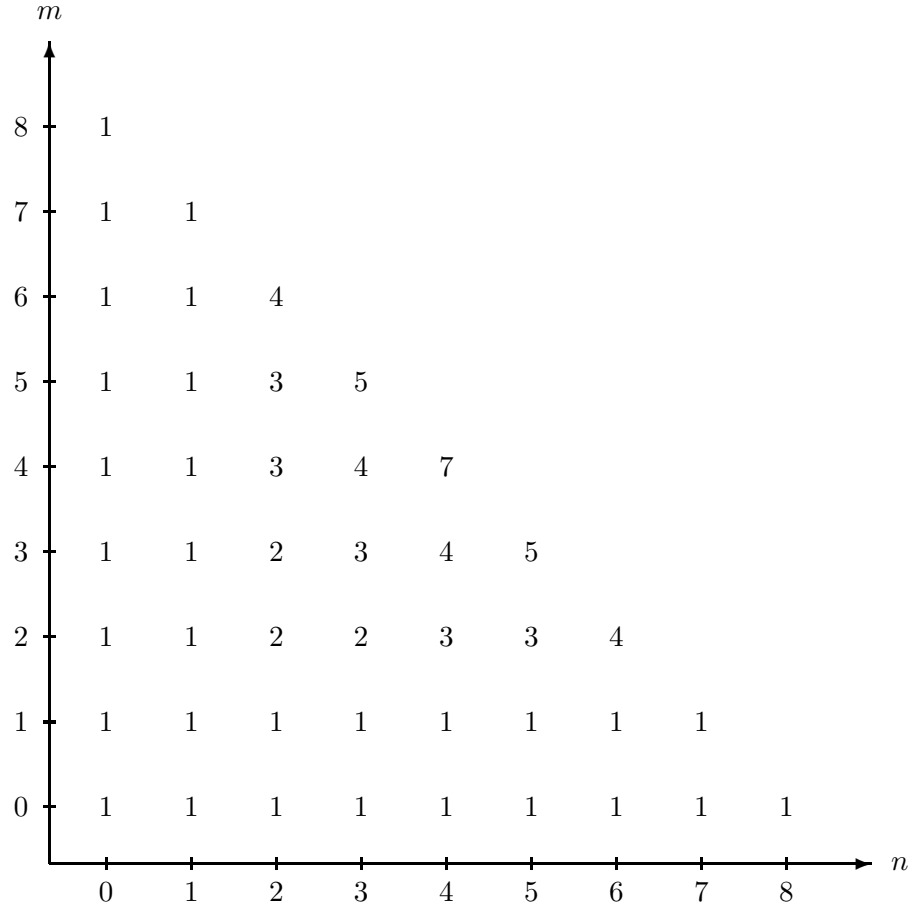


Figure 1: The degeneracies of operators L_{mn} for the points (m, n) with $m + n \leq 8$.

degeneracies, *i.e.* from the number of independent operators L_{mn} for given (m, n) . In fig.1 a list of degeneracies is given for the lowest values of $m + n$.

I will here give some of the expressions for the generators for the lowest values of $m + n$. Since they all are operators of the one-particle form I may for simplicity suppress the particle index and the summation over this index. It is also sufficient to list the generators only for $m \geq n$ due to the Hermiticity relation $L_{nm} = L_{mn}^\dagger$. For $m + n \leq 3$ only the maximal spin multiplets exist, with values running from 0 to 3/2. The expressions are

$$\begin{aligned}
s = 0 : \quad & L_{00} = 1 \\
s = \frac{1}{2} : \quad & L_{10} = a^\dagger \\
s = 1 : \quad & L_{20} = a^{\dagger 2} \quad L_{11} = \frac{1}{2}(a^\dagger a + a a^\dagger) \\
s = \frac{3}{2} : \quad & L_{30} = a^{\dagger 3} \quad L_{21} = \frac{1}{2}(a^{\dagger 2} a + a a^{\dagger 2})
\end{aligned} \tag{37}$$

For $m + n = 4$ there are two multiplets, of spin 2 and 0. The spin 2 operators are

$$L_{40}^2 = a^{\dagger 4} \quad L_{31}^2 = \frac{1}{2}(a^{\dagger 3}a + aa^{\dagger 3}) \quad L_{22}^2 = \frac{1}{6}(a^{\dagger 2}a^2 + a^{\dagger}aa^{\dagger}a + a^{\dagger}a^2a^{\dagger} + \text{sym}) \quad (38)$$

where *sym* means the symmetric expression by right-left reflection of the products. The spin 0 operator is

$$L_{22}^0 = L_{31} = a^{\dagger 2}a^2 + a^{\dagger}aa^{\dagger}a + \text{sym} \quad (39)$$

The way these and other operators have been found is to construct the operators level by level in the variable $m + n$. One operator, *e.g.* L_{30} , may be used to step up to the next level and the operators L_{20} and L_{02} may be used to complete the spin multiplets. With the help of these three operators all the others can be generated. Identities between the operators can be found by reducing all operators to a standard form by use of the modified Heisenberg commutation relation (24).

8 Other formulations of the algebra

The expressions for the observables L_{mn} given above refer to the representation of the algebra in the Calogero model. I will now sketch how the algebra can be expressed in a more general way [17]. To do so let us consider the the adjoint representation, with each element expressed as a product of a and a^{\dagger} . The observables L_{mn} then act on products of a and a^{\dagger} and map them into new products. We may think of the following commutators as the fundamental ones,

$$\begin{aligned} [L_{m0}, a^{\dagger}] &= 0 & [L_{m0}, a] &= -ma^{\dagger(m-1)} \\ [L_{0n}, a^{\dagger}] &= na^{n-1} & [L_{0n}, a] &= 0 \end{aligned} \quad (40)$$

From these expressions the action of a general element L_{mn} , which is defined by repeated commutations of L_{m0} and L_{0n} , can be deduced. The only additional input needed is that an observable L_{mn} , when acting on a product of a and a^{\dagger} , gives a sum of products where the observable acts on each element in the product, *e.g.*

$$[L_{mn}, aa^{\dagger}a\dots] = [L_{mn}, a]a^{\dagger}a + a[L_{mn}, a^{\dagger}]a + aa^{\dagger}[L_{mn}, a] + \dots \quad (41)$$

This trivially follows from the fact that the observable act on the product by commutation.

Note that in this construction we do not need to establish the explicit expressions for the observables in terms of a and a^{\dagger} . Therefore the question of reordering of operators does not enter. Also note that the result of acting with an observable on a product of a and a^{\dagger} is an expression with a unique ordering of operators. Let us therefore forget about the Calogero model and *define* the algebra of observables L_{mn} by the relations (40) and (41). We assume no commutation rule for a and a^{\dagger} , so different orderings have to be considered as distinct. We note that (41) has the form of Leibniz rule so the observables L_{mn} can be interpreted as differential operators acting on the products (the non-commutative ring) generated by a and a^{\dagger} . Clearly these differential operators define an infinite-dimensional algebra.

The claim is now that the algebra defined in this more abstract way is the same as the one defined by use of the operators of the Calogero model. This may seem strange since

we have not taken into account the identities which earlier created problems for an explicit construction of the algebra. But the identities are in fact there, although in a somewhat different form. An element $I_{mn} = L_{mn} - L'_{mn}$, in the new definition of the algebra, is identified with zero if it commutes with all products of a and a^\dagger . A sufficient condition for this to happen is that

$$[I_{mn}, a] = [I_{mn}, a^\dagger] = 0 \quad (42)$$

Therefore the identities are automatically taken care of when the operators L_{mn} are defined by their action on the products of a and a^\dagger .

To see the correspondance with the algebra of the Calogero model more explicitly, we note that the defining relations (40) there can be represented in the following form

$$\begin{aligned} [\sum_i a_i^{\dagger m}, a_j^\dagger] &= 0 & [\sum_i a_i^{\dagger m}, a_j] &= -m a_j^{\dagger(m-1)} \\ [\sum_i a_i^n, a_j^\dagger] &= n a_j^{n-1} & [\sum_i a_i^n, a_j] &= 0 \end{aligned} \quad (43)$$

Note that even if there is no summation over the particle index j , these relations are independent of the statistics parameter ν . This implies that any observable L_{mn} when commuted with a product of a_j and a_j^\dagger gives a sum of products of these operators – only referring to particle j – which is independent of ν . In particular this is true for the Calogero Hamiltonian (see (29)), and this fact has already been exploited in [21] to construct eigenstates and eigenvalues of the Hamiltonian. For products of a_j and a_j^\dagger , with no sum over j , a re-ordereding of operators by use of the modified Heisenberg commutation relations will give terms which depends on ν . It is only after the summation over the particle index that the ν -independent identities appear. The correspondence between the operators a and a^\dagger in (40) and the un-summed Calogero operators therefore gives an explanation why these identities should not be implemented.

As already mentioned we may view the observables L_{mn} as linear differential operators acting on products of a and a^\dagger . A general element of the algebra can be written in the form

$$L = P(a, a^\dagger) \frac{\partial}{\partial a^\dagger} + Q(a, a^\dagger) \frac{\partial}{\partial a} \quad (44)$$

where P and Q are polynomials in a and a^\dagger . This formulation of the observables in fact introduces a simplification in the construction of the algebra. Commutators can be calculated by the use of these expressions, and there is no ambiguity in the resulting expressions due to identities between reordered products of a and a^\dagger . However, I will now discuss another representation, in terms of matrices, which gives an even simpler way to do the construction.

9 The matrix model

When discussing the algebra $W_{1+\infty}$ of bosons and fermions, the presence of higher order terms in \hbar was pointed out. For the algebra of observables discussed here such higher order terms are not present. In the construction of the algebra by use of the operators of the Calogero model this can be understood by the way the commutators are evaluated, namely by applying the (modified) Heisenberg commutation rule only once in the evaluation of a commutator.

It was not used repeatedly to bring the expression into a standard form; instead expressions with different ordering of operators were accepted as different elements of the algebra.

The lack of higher order terms in \hbar indicates that the same algebra is present also at the classical level. This is in fact the case, and the defining relations (24) are clearly satisfied for a classical N -particle system if the commutators are replaced by Poisson brackets, and the quantum variables a and a^\dagger are replaced by their classical counterparts (or by x and p). But the algebra is not represented faithfully in this way. Since x and p commute, there is no distinction between expressions differing by reordering these variables. If the algebra should be represented faithfully at the classical level we have to consider a system where the phase space variables do not commute at the classical level.

A classical system where x and p do not commute may seem rather unconventional. However, the matrix models are exactly of this type. In these models the coordinate x as well as the momentum p are matrix-valued phase-space variables. Even if each matrix element of x commutes with each matrix element of p , they do not commute as matrices. For finite N there will exist certain relations between reordered (matrix) products of x and p , but in the limit $N \rightarrow \infty$ there will be no relations surviving between reordered products.

The fundamental Poisson brackets for the matrices x and p are

$$\{x_{ij}, p_{kl}\}_{pb} = \delta_{il}\delta_{jk} \quad (45)$$

and for general functions of x and p they get the form

$$\{C, C'\}_{pb} = \sum_{ij} \left[\frac{\partial C}{\partial x_{ij}} \frac{\partial C'}{\partial p_{ji}} - \frac{\partial C}{\partial p_{ji}} \frac{\partial C'}{\partial x_{ij}} \right] \quad (46)$$

By use of these bracket relations it is straight forward to demonstrate that the defining relations of the algebra (24) are satisfied. Expressed in terms of x and p (rather than a and a^\dagger) they have the form

$$\begin{aligned} \{Tr p^m, p\}_{pb} &= 0 & \{Tr p^m, x\}_{pb} &= -mx^{m-1} \\ \{Tr x^n, p\}_{pb} &= nx^{n-1} & \{Tr x^n, x\}_{pb} &= 0 \end{aligned} \quad (47)$$

Comparing with the expressions of the Calogero model, we note that the products with no sum over the particle index j in the present case correspond to the matrix products of x and p (without taking the trace) whereas the products with summation over the particle index correspond to the trace of the matrix products. The elements of the algebra then correspond to such traced matrix products.

The matrix formulation now can be used to construct the algebra step by step in the same way as discussed for the Calogero model. The Poisson brackets are evaluated by use of (46), and in the evaluation the order of the variables of the matrix products have to be respected. This matrix representation is simpler for two reasons. Firstly, the cyclic property of the trace means that there are fewer products of x and p to care about. Secondly, there are no additional (hidden) identities between products with different ordering of x and p to worry about. It is of interest to note that no matrix multiplication has actually to be performed. The evaluation of Poisson brackets can be formalized in a diagrammatic form where a matrix product is represented by an ordered string of two different objects (x and p). The untraced

product is an open string and the traced product is a closed string. The Poisson bracket is implemented as a rule for merging two of these strings.

To construct the algebra in the matrix representation, the rules for how to evaluate Poisson brackets on a phase space of non-commuting x and p is all we need. It is nevertheless of interest to be more specific about the matrix model in order to see the correspondence with the original system of N identical particles in one dimension [24]. Let me then assume x and p both to be Hermitian $N \times N$ matrices with a free particle Hamiltonian of the form

$$H = \frac{1}{2} \text{Tr} p^2 \quad (48)$$

The time evolution in matrix form is simply

$$x = x_0 + vt \quad (49)$$

with x_0 and v as constant matrices. The position and velocity do not have to commute as matrices, but the commutator is a constant of motion

$$K = i[x, \dot{x}] = i[x_0, v] \quad (50)$$

The Hamiltonian as well as all other elements of the algebra of observables, which can be written as the trace of products of x and p , are invariant under $SU(N)$ transformations of the form

$$x \rightarrow UxU^\dagger \quad p \rightarrow UpU^\dagger \quad (51)$$

In particular this is the case for the invariant coordinates $x^{(k)} = \text{Tr}(x^k)$ which determine the eigenvalues of the matrix x . These eigenvalues we may view as the coordinates of a one-dimensional N -particle system. The $SU(N)$ group may be regarded as an extension of the permutation group for this N -particle system and it allows the positions of the N particles to be continuously permuted. This continuous transformation involves the non-invariant parts of x and the extension of the permutation group to $SU(N)$ can be seen as a compensation for the fact that more degrees of freedom are present in the matrix model than in the N -particle system.

A separation of the $SU(N)$ invariant and the non-invariant coordinates can be made explicit by a (time-dependent) diagonalization of x ,

$$x = U^\dagger x_d U \quad (52)$$

Here x_d is a diagonal matrix which depends on the (invariant) eigenvalues, while U depends on a set of (generalized) angular variables which are non-invariant under the $SU(N)$ transformations. Expressed in terms of x_d and U the Hamiltonian gets the form

$$H = \frac{1}{2} \text{Tr}(\dot{x}_d^2 + J^2) \quad (53)$$

where

$$J = [x_d, \dot{U}U^\dagger] \quad (54)$$

We note that if the angular velocities $\dot{U}U^\dagger$ can be traded with a set of conserved quantities (generalized angular momenta), one may be able to write H as a function only of the variables x_d , their time-derivative and these conserved quantities. The Hamiltonian then describes a set of N interacting particles with $\frac{1}{2}\text{Tr}(J^2)$ as the potential. This reduction of the number of variables clearly is analogous to the reduction to the radial variable for a rotationally symmetric one-body problem.

An explicit construction of this type can be given [24]. Let us choose a basis where the coordinate x_0 at time $t = 0$ is diagonal and give the system an initial velocity v with non-diagonal matrix elements

$$v_{ij} = -i \frac{l}{x_{0i} - x_{0j}}, \quad i \neq j \quad (55)$$

with x_{0i} as the eigenvalues of x_0 . (The diagonal matrix elements of v are arbitrary.) The conserved quantity K then has non-diagonal matrix elements which all are equal,

$$K_{ij} = l, \quad i \neq j \quad (56)$$

while the diagonal matrix elements necessarily vanish. We have the following relation between K and J ,

$$UKU^\dagger = i[x_d, J] \quad (57)$$

which in component form gives

$$l \sum_{m \neq n} U_{im} U_{jn}^* = i(x_i - x_j) J_{ij} \quad (58)$$

with x_i as the eigenvalues of x . From the diagonal case $i = j$ and the unitarity of U we deduce

$$|\sum_m U_{im}|^2 = 1 \quad (59)$$

which, when re-inserted in (58) gives

$$J_{ij} = -i l \frac{\alpha_i \alpha_j^*}{x_i - x_j}, \quad i \neq j \quad (60)$$

where $\alpha_i = \sum_m U_{im}$ is a phase factor. This gives the following expression for the Hamiltonian

$$H = \frac{1}{2} \left(\sum_i \dot{x}_i^2 + \sum_{i \neq j} \frac{l^2}{(x_i - x_j)^2} \right) \quad (61)$$

which is the same as that of a (classical) system of N identical particles interacting through a $1/x^2$ two-body potential.

The similarity between the interacting N -particle system and a free system now becomes clearer. The system interacting with a two-body $1/x^2$ potential can be derived from a free system with more degrees of freedom, namely the $N \times N$ matrix model, and the potential arises in a similar way as a centrifugal potential of a free particle in 2 or 3 dimensions when this is described in radial coordinates. The correspondence between the classical systems discussed here carries over to the quantized systems, but then with restrictions on the possible values of l due to the compactness of the $SU(N)$ group.

10 Concluding remarks

The purpose of the work which has been reviewed here is to examine generalizations of quantum statistics from the point of view of Heisenberg quantization. This approach is interpreted as a problem of finding representations of a fundamental algebra of observables for a system of N identical particles. For 2 particles in one dimension the algebra is identified as $su(1,1)$, and the representations are readily found. There are generalizations of the Bose and Fermi cases which can be expressed in terms of a statistical $1/x^2$ interaction. For the N particle system the statistical interaction is assumed to be a two-body interaction of the same form. In this way one is lead to consider the Calogero model as describing a system with generalized statistics.

Much of the discussion I have given concerns the question of whether a fundamental algebra of observables can be identified for the N -particle system, an algebra which replaces $su(1,1)$ for two particles. Since the Calogero model is assumed to describe a system with generalized statistics, the operators of this model can be used to construct the algebra. A main result is that such an algebra, which is independent of the (statistics) parameter, can indeed be constructed. The algebra is infinite-dimensional, and even if a closed formulation of the algebra has not been found, a method for constructing a basis for the algebra step by step can be given. An interesting point which has been emphasized is that a more general formulation of the algebra can be given than the one given in terms of operators of the Calogero model. This general formulation can be used to demonstrate the presence of the same algebra also in other systems, in particular in the Hermitian matrix model.

There remain several unsolved problems concerning the algebra of observables which has been discussed here. A main mathematical problem is to identify more fully the structure of the algebra. Another question concerns the possibility of finding other (physically) interesting representations. In addition to these questions about the mathematical structure of the algebra, one has the important question of whether there are physical systems where this structure appear in a natural way. When the algebra of the 2-particle system was originally introduced it was applied to the system of vortices in a superfluid helium film [13]. It may be questionable whether such vortices are good candidates for the application of ideas of (one-dimensional) statistics, but there exist other vortex-like excitations which may be. Anyons in a strong magnetic field have been demonstrated to satisfy this type of statistics [25], and the correspondence to the Laughlin quasi-particle excitations suggest that the algebra discussed here may be relevant for descriptions of the quantum Hall system. However, my discussion has aimed mainly at demonstrating how, in the framework of Heisenberg quantization, the algebra of observables for a system of identical particles in one dimension appears as unique mathematical structure. The other interesting questions about further investigations and about possible applications I will have to leave as open questions for future research.

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